THE GROUPS WHICH ARE GENERATED BY TWO OPERATORS OF ORDERS TWO AND FOUR RESPECTIVELY WHOSE COMMUTATOR IS OF ORDER TWO.

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We shall first give two general theorems relating to commutators[1] which will be used in what follows.

Theorem I. If a commutator is commutative with one of its elements the order of the commutator divides the order of this element.

Theorem II. The smallest invariant subgroup which contains the commutator of two operators includes the commutator subgroup of the group generated by these operators.

To prove the former of these two theorems, it is only necessary to observe that if $c = s_1^{-1}s_2^{-1}s_1s_2$ is commutative with s_1 or s_2 it is also commutative with $s_2^{-1}s_1s_2$ or $s_1^{-1}s_2s_1$, respectively. Hence s_1 or s_2 would be commutative with $s_2^{-1}s_1s_2$ or $s_1^{-1}s_2s_1$, respectively. The proof of the theorem follows now from the facts that the order of the product of two commutative operators divides the least common multiple of their orders, and that the orders of the two factors in the present case are the same. The proof of the second theorem follows from the fact that the two operators in question would correspond to commutative operators in the quotient group with respect to the given invariant subgroup. In particular we have the corollary, *If two operators generate a group the conjugates of their commutator generate the commutator subgroup.*

Let s_1, s_2 be any two operators of orders 2 and 4, respectively, which satisfy the conditions

$$s_1^2 = 1, \quad s_2^4 = 1, \quad s_1s_2^2s_1s_2 = s_2^3s_1s_2s_1,$$

[1] The principal known theorems relating to commutators are found in two articles published in the *Bulletin of the American Mathematical Society*, vol. 4, p. 135 and vol. 6, p. 105.

and consider the four operators

$$s_1s_2^3s_1s_2, \quad s_1s_2s_1s_2^3, \quad s_2^3s_1s_2^3s_1s_2^2, \quad s_2^2s_1s_2^3s_1s_2^3.$$

The common product of the two commutators and of the other two factors is $s_2^3s_1s_2^2s_1s_2^3$. Hence each of these four factors transforms this common product into its inverse. Moreover, the common product of the first and third and of the second and fourth of the four given operators is $s_1s_2^2s_1s_2^2$, and each of these operators transforms this common product into its inverse. From this it follows that the two operators

$$s_2^3s_1s_2^2s_1s_2^3 \text{ and } s_1s_2^2s_1s_2^2$$

are commutative and that the group generated by them is either cycle or the direct product of two cyclic groups. This abelian group, together with $s_1s_2^3s_1s_2$, generates a group (H), which involves $s_1s_2s_1s_2^3$, $s_2^3s_1s_2^3s_1s_2^2$, $s_2^2s_1s_2^3s_1s_2^3$. The order of H cannot be more than twice that of the given abelian subgroup, and H is invariant under the group (G) generated by s_1 and s_2, since s_2 transforms the four generators of H among themselves and s_1 transforms the first two of them into themselves, while

$$s_1s_2^3s_1s_2^3s_1s_2^2s_1 = s_1s_2^3s_1s_2 \cdot s_2^2s_1s_2^3s_1s_2^3 \cdot s_2s_1s_2^3s_1.$$

From theorem II we conclude that H is the commutator subgroup of G and we have the preliminary theorem: If a group is generated by two operators of orders 2 and 4, respectively, whose commutator is of order 2, the commutator subgroup must be of one of the following two types: the cyclic group of order 2, the group obtained by extending a cyclic group or the direct product of two cyclic groups by means of an operator of order 2 which transforms each operator of this direct product into its inverse.

The group generated by H and s_1 is invariant under s_2, since it contains $s_2^3s_1s_2$. Hence the order of G cannot exceed eight times that of H. As the operators of odd order in G (if such operators occur) generate a characteristic subgroup of its commutator subgroup the order of the commutator quotient group is either 4 or 8; that is, the order of G is either four or eight times that of H.

The octic group is clearly the group of smallest order which contains two operators of orders 2 and 4, respectively, whose commutator is of order 2. In this case the commutator subgroup is cyclic and the commutator quotient group is the four-group. There is one group of order 16 which is generated by two operators of orders 2 and 4, respectively, whose commutator is of order 2. This may be defined as the group of order 16 which involves the abelian subgroup of type (1, 1, 1) and 4 cyclic non-invariant subgroups of order 4. In this case the commutator subgroup is again cyclic and the commutator quotient group is of type (2, 1). Since this group of order 16 has a (2, 1) isomorphism with the octic group, and since the operators of odd order in G generate a characteristic subgroup, it follows that *every group which is generated by two operators of orders* 2 *and* 4, *respectively, whose commutator is of order* 2, *has the octic group for a quotient group, and its commutator quotient group is either of type* (1, 1) *or of type* (2, 1).

Suppose that the commutator quotient group of G is of order 4. If the order of G were divisible by 16 it would have the group of order 16 considered in the preceding paragraph as a quotient group. This is impossible since the commutator quotient group of the latter is of type (2, 1). That is, *if the commutator quotient group of G is of order* 4 *the order of G is* 8 *m, m being an odd number.*

If H contains an invariant cyclic subgroup whose order exceeds 2, the operators of H, which are commutative with a generator of this subgroup, constitute an invariant subgroup of G. This invariant subgroup does not include $s_1s_2^3s_1s_2$, and hence it does not include H. Since the group of isomorphisms of a cyclic group is abelian and every invariant subgroup which gives rise to an abelian quotient group includes H, the assumption that H contains an invariant cyclic subgroup whose order exceeds 2 has led to a contradiction. From this it follows that the groups generated by $s_2^3s_1s_2^2s_1s_2^3$ and $s_1s_2^2s_1s_2^2$, respectively, have at most 2 common operators and that these cyclic groups are either of the same order or the order of one is twice that of the other. In particular, if the order of G is divisible by an odd prime number p, the highest power of p which divides this order has an even index. The main results which have been obtained may be expressed as follows: *If a group is generated by two operators of orders 2 and 4, respectively, whose commutator is of order 2, the commutator subgroup is either the cyclic group of order 2, or it may be constructed by extending the direct product of two cyclic groups, involving the same odd factors, by means of an operator of order 2, which transforms each operator of this direct product into its inverse.*

It should be observed that the Sylow subgroups whose orders are powers of 2 in the factors of the direct product of the preceding theorem are either of the same order or one of these orders is twice that of the other. In particular, this direct product may be of order 2. In this case H is the four-group. When H is abelian its order can clearly not exceed 8, and when it involves operators of odd order its order cannot be less than 18. The octic group and the given group of order 16 are evidently the only possible groups when H is cyclic and the substitution group of order 64 and degree 8 which may be generated by $s_1 = ae$, $s_2 = abcd \cdot efgh$ is a group in which G has its largest possible value when H is abelian. As an instance of a G which involves operators of odd order we may mention the transitive group of degree 6 and of order 72. This may be generated by $s_1 = ae$ and $s_2 = abcd \cdot ef$, and is the smallest possible group involving operators of odd order, which may be generated by two operators of orders 2 and 4, respectively, whose commutator is of order 2.

Let s_2 be a substitution of degree $2p$, p being any odd prime number, involving only one transposition. Representing s_2 by $a_1a_2a_3a_4 \cdot b_1b_2b_3b_4 \cdot \ldots l_1l_2l_3l_4 \cdot mn$ and s_1 by $a_3b_1 \cdot b_3c_1 \cdot \ldots l_3m$, it is clear that s_1, s_2 generate a transitive group of degree $2p$. As the order of H is divisible by p^2 and as it contains a direct product of two cyclic groups whose orders are divisible by p, this direct product must be of order p^2. The order of H is therefore $2p^2$ and H is obtained by establishing a (p, p) correspondence between two dihedral groups of order $2p$. Since $s_2{}^2$ is not commutative with any substitution in the direct product of order p^2 contained in H besides the identity, it has p^2 conjugates under this direct product and hence transforms each one of its substitutions into its inverse. From this it follows that s_2^2 and this direct product generate H, and hence the order of G is $8p^2$. We have thus arrived at an interesting infinite system of transitive groups of degree $2p$ and of order $8p^2$, such that each group is generated by two substitutions of orders 4 and 2, respectively, whose commutator is of order 2.

As an indirect result of the preceding paragraph we have that the group of isomorphisms of the abelian group of order p^2 and type (1, 1) must always include the octic group. As the abelian group of order p^m, $m > 2$, and of type (1, 1, 1, ...) can be made simply isomorphic with itself after any correspondence has been established between two of its subgroups, it follows from the above that *the octic group is a subgroup of the group of isomorphisms of the abelian group of order* p^m, $m > 1$, and of type (1, 1, 1,...), *p being any odd prime number.* In particular, this group of isomorphisms contains two operators of orders 2 and 4, respectively, whose commutator is of order 2.

The main results of the preceding paragraphs may be stated as follows: If G is generated by two operators of orders 2 and 4, respectively, whose commutator is of order 2, then G is solvable and has an $(\alpha, 1)$ isomorphism with the octic group. The commutator quotient group of G is of order 4 or 8. When this order is 4 the order of G is $8m$, m being an odd number, and vice versa. The highest power of an odd prime factor of the order of G has an even index and the Sylow subgroup of this order is the direct product of two cyclic subgroups of the same order. When the commutator subgroup of G is cyclic G is either of order 8 or of order 16 and vice versa. The only other abelian commutator subgroups which may occur in G are the four-group and the abelian group of order 8 and of type (1, 1, 1). The other possible commutator subgroups of G may be obtained by extending the direct product of two cyclic groups by means of an operator of order 2 which transforms each operator of this direct product into its inverse. A cyclic subgroup whose order exceeds 2 which is contained in this direct product is not invariant under G.

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